Question No: 1

i) Give a linear differential equation relating attenuation of incident radiation, $\phi$ Wm$^{-2}$, with distance $z$ through a medium of extinction coefficient $k$ m$^{-1}$. [10 marks]

ii) Derive the general solution of this equation. Show your working. [20 marks]

iii) Solar radiation incident on the Earth’s surface is measured as follows for 3 solar zenith angles $\theta$ (angle from the vertical):

<table>
<thead>
<tr>
<th>Solar zenith angle, $\theta$</th>
<th>$\phi(z)$ Wm$^{-2}$</th>
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<tbody>
<tr>
<td>30°</td>
<td>871</td>
</tr>
<tr>
<td>45°</td>
<td>785</td>
</tr>
<tr>
<td>60°</td>
<td>620</td>
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Use these values, and your expression derived in part ii) to show that the extinction coefficient $k$, is approximately 0.4 and that the solar constant (i.e. $\phi$ at $z = 0$ at the top of the atmosphere) is $\sim$1385 Wm$^{-2}$. [70 marks]

Model Answer:

i) $d(\phi)/dz$ proportional to $-\phi$ and $d(\phi)/dz = -k \phi$

ii) Separate and integrate i.e. 

$$\int \frac{d\phi}{\phi} = -k \int dz \Rightarrow \ln(\phi) = -kz + C, \phi(z) = \phi_0 e^{-kz}$$

iii) Take logs of both sides so $\log(\phi) = \log(\phi_0) - kz$. Plot these numbers on a graph to show slope is $-k$; intercept $(0 = 0, m = 1)$ = 7.23. So $\phi_0 = e^{(7.23)} = 1386$ Wm$^{-2}$. 
Module Code: GEOGG141
Module Title: Analytical and Numerical Methods
Contributor: M. Disney

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Question

i) State Bayes’ Theorem, explaining each of the separate terms in the expression. [20 marks]

ii) Suppose a screening test for a particular cancer is 99% accurate in producing a positive diagnosis of having cancer when cancer is present, and 98% accurate in producing a negative diagnosis i.e. of being cancer-free when cancer is not present (i.e. we get twice as many false negatives as false positives). Suppose also that 5% of the general population have this particular cancer. Given these figures, is this test for cancer useful? Explain your answer. [80 marks]

Model Answer:

i) Bayes’ Theorem - either give the rule as:

\[
P(H|D) \propto P(H|D) \times P(H)
\]

or could give full definition i.e.

\[
P(H|D, I) = \frac{P(D|H, I) \times P(H, I)}{P(D|I)}
\]

where \(P(H|D, I)\) is posterior prob. of hypothesis \(H\) being true, given data \(D\), and background conditioning information \(I\); \(P(D|H, I)\) is the likelihood function i.e. prob. that data \(D\) would be observed if \(H\) is true; \(P(H, I)\) is the prior prob. i.e. prob of hypothesis \(H\) being true before measurement of \(D\); \(P(D|I)\) is the evidence i.e. prob. of data \(D\) being observed.

ii) Use calculation from i) i.e. \(P(\text{cancer|+ve}) = P(\text{+ve|cancer}) \times P(\text{cancer}) / [P(\text{+ve|cancer})P(\text{cancer}) + P(\text{+ve|no cancer})P(\text{no cancer})]\) = 0.99 x 0.005 / [(0.99 x 0.005 + 0.02 x 0.995)] = 20%. There is a 20% chance that you have cancer if you test +ve i.e. the chance of you being healthy are 80%. This is not an effective test. The reason is that the false positive rate (.02 x 0.995 = 2%) is more than 4 times greater than the true positive rate (0.99 x 0.005 = 0.495%). In a large number of patients, there will be a much greater number of healthy patients than cancer sufferers, providing a large number of possibilities of false positives, even though the probabilities of false positive are small.
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**Special Instructions**

**Question No:** 3

**Question**

i. Determine whether the following models are linear or non-linear, giving reasons in each case:

a. \( y = ax^2 + \frac{bx^4}{4} + cx^5 \) where a, b, c are unknown model parameters.  
[5 marks]

b. \( y = asin(cx) + bcos(cx) \) where a, b, and c are unknown model parameters.  
[5 marks]

c. \( y = a_0 + \sum_{i=1}^{n}[asin(x_i) + bcos(x_i)] \) where a\textsubscript{i}, b\textsubscript{i} are unknown model parameters.  
[5 marks]

d. \( y = a_0e^{-a_1x} \) where a\textsubscript{i} are unknown model parameters.  
[5 marks]

ii. Describe TWO methods of non-linear model inversion, outlining the advantages and disadvantages of each method and any assumptions required for applying it.  
[80 marks ]

**Model Answer:**

i. a is linear – simple polynomial in x; b) the dependence of c in each term means it is non-linear; c) linear, a sum of terms each multiplied by a model parameter; non-linear – exponential function and model parameter in exponent.

ii. Any one of Powell, Brent, simplex, simulated annealing, LUT or one of derivative-based methods eg BFGS. For a good answer expect clear description, figures, issues as to how to implement and an example of applying it.
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<tr>
<td>Question</td>
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<tr>
<td>i.</td>
<td>Briefly outline the advantages and disadvantages of using a linear model as opposed to a non-linear one. [20 marks].</td>
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<tr>
<td>ii.</td>
<td>Describe ONE application of linear kernel-driven BRDF models in describing surface reflectance. Your answer should describe the model formulation, outlining the assumptions underlying the linear BRDF model approach. [80 marks].</td>
</tr>
<tr>
<td>Model Answer:</td>
<td></td>
</tr>
<tr>
<td>i.</td>
<td>Primary advantage is can use linear algebra techniques to reduce problem to one of matrix inversion, which is well-understood and can be done rapidly and efficiently in any coding environment. Easier to handle uncertainties constrained inversion. Disadvantage is cannot be as flexible as a non-linear model.</td>
</tr>
<tr>
<td>ii.</td>
<td>Could be any one of change detection (burns), albedo, or angular normalisation. In either case would need to outline kernel-driven approach, shapes and terms and then how the models can be applied. Good answer would describe RT theory behind the kernels including assumptions (GO - sscattering, no overlapping shadows, crown = ground), VOL (mscatt isotropic, lambertian soil).</td>
</tr>
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Question

i. Considering the four time series shown below, describe the properties of each time series. [5 marks]

![Time Series Images]

ii. For time series “a” (above), describe a method that you would use to decompose the time series into its constituent parts. Justify your choice of method. [10 marks]

iii. What problems would time series “d” pose for a statistical analysis? [5 marks]

iv. Write down the equation for an autoregressive process of order 3 for the series $x_t$. 
v. Given the following notation SARIMA(2,1,2)(0,1,1)\(_{12}\), briefly describe the type of model fitted and the orders of the various processes involved.

vi. Consider a time series of length \( n = 100 \), modelled by generalised least squares assuming an AR(1) process for the correlation matrix of the residuals. The estimated AR(1) parameter, \( \rho \), is 0.7.

Complete the following portion of the correlation matrix. The last row and column should be written for the \( n=100^{th} \) observation:

\[
P = \begin{pmatrix} \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & 1 & \cdot & \cdots & \cdot \\ \cdot & \cdot & 1 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdots & 1 \end{pmatrix}
\]

vii. Choosing one of the times series “a”, “c”, or “d” (not time series “b”!) shown in the figure above, describe the method(s) you choose to model the properties of the time series. Where appropriate, show the algebraic equation for your any statistical model, and justify your choice of technique(s).

Model Answer:

i. a) Trend with additive seasonal component – trend possibly non-linear. b) Aperiodic cyclic variation – NOT seasonal variation or period cyclic variation. c) Non-linear trend. d) Trend with multiplicative seasonal component.

ii. Several options were taught (classical and Loess-based decompositions, explicit modelling of the trend and seasonal component via a regression-based). Classical decomposition is not competitive here as the seasonal component changes in magnitude in time yet this method assumes it is constant. Likewise, a linear regression is inappropriate as the trend looks non-linear and there is appreciable temporal autocorrelation. Generalised least squares can deal with the temporal autocorrelation but not so easily with the non-linear trend. Loess-based decomposition or an additive model with a time series process for residuals would be the best choices to get full marks. For justification, student should show that they appreciate that the trend is not linear and the seasonal component may not be regular. Hence the classical decomposition is more than likely to be the poorest in terms of extracting the seasonal component. The linear regression techniques (least squares, GLS) are likewise not recommended.

iii. The main issue is the multiplicative seasonal component that needs to be made additive. In a classical time series analysis (e.g. SARIMA) need to control for this non-constant variance via e.g. a log transformation, plus need to make the series stationary via differencing to remove trend. Additional issue is checking that the residuals variance is also non-constant. If it isn’t, the transformation will not be effective. For full marks must mention the issue of whether errors are multiplicative as well as the seasonal component, plus making the series stationary. If student chooses to discuss from a regression point of view, the temporal dependence needs to be mentioned and controlled for and the non-constant variance assumption of regression accounted for. For full marks student needs to mention the temporal dependence issue (violation of independence) and the non-constant variance issue, and problem of choosing an appropriate correlation structure for residuals.
iv. \( x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \alpha_3 x_{t-3} + z_t \)
Could also write as a summation of terms over 1, 2, 3:
\[
x_t = \sum_{i=1}^{3} (\alpha_i x_{t-i}) + z_t
\]
For full marks, stating the distributional assumptions of \( z_t \) is required either in words (\( z_t \) is a white noise or Gaussian process) or via notation \( z_t \sim N(0, \sigma^2) \).
Note that the parameters \( \alpha \) can be written as any symbol.

v. Notation describes a seasonal autoregressive integrated moving average model. The model involves first order differencing, a second order autoregressive process (AR(2)), and a second order moving average process (MA(2)), plus first-order seasonal differencing and a first order seasonal moving average process (MA(1) or SMA(1)). For full marks, the student should note that the order of seasonality is 12 (i.e. monthly data).
\[
P = \begin{pmatrix}
1 & 0.7 & 0.7^2 & \cdots & 0.7^{n-1} \\
0.7 & 1 & 0.7 & \cdots & 0.7^{n-2} \\
0.7^2 & 0.7 & 1 & \cdots & 0.7^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0.7^{n-1} & 0.7^{n-2} & 0.7^{n-3} & \cdots & 1
\end{pmatrix}
\]

vi. would be acceptable, but for all 20 marks compute the terms and note that it is n-1, n-2 in final row/column of the matrix
\[
P = \begin{pmatrix}
1 & 0.7 & 0.49 & \cdots & 4.6e^{-16} \\
0.7 & 1 & 0.7 & \cdots & 6.6e^{-16} \\
0.49 & 0.7 & 1 & \cdots & 9.4e^{-16} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
4.6e^{-16} & 6.6e^{-16} & 9.4e^{-16} & \cdots & 1
\end{pmatrix}
\]

vii. All could be modelled quite easily using (S)ARIMA model, c) is annual data so an ARIMA would be appropriate. For a) students did this as an example in the lecture and the computer class, so should be able to recall that MA terms required for the main and seasonal parts of the model. First order and first-order seasonal differencing will be required as there is a trend and seasonal pattern that needs removing to bring the series into stationarity (for “c” only annual data so no seasonal part to the model). If choosing SARIMA, then the ACF and pACF of the differenced and seasonally differenced series will help choose which types of time series processes would be required (i.e. AR or MA terms). Fit a series of candidate SARIMA models and select among the best models using BIC. Must mention model checking of residuals from the fitted model; looking for no trend/pattern in residuals, non-constant variance of residuals, and no residual autocorrelation. For “d” student must consider the transformation – SARIMA should be fitted to the log of the series. If the student chooses to model via a regression then they must consider whether the trend is linear or not. For linear trend GLS is appropriate, the additive model approach is appropriate for the non-linear trend. Also must consider fitting appropriate correlation process for residuals. I gave the students a guide to fitting these models, so they should be able to go through process of getting fixed effects part specified (trend + season or trend only), then fit model with and without correlation structure, decide on need for correlation structure using a likelihood ratio test (LRT), then return to fixed effects and decide if there is a trend or not using LRT. Expect to see appreciation of non-constant variance in “d”, sufficient modelling of autocorrelation structure in residuals of models – checked via ACF and pACF on residuals, and if modelling using additive model, the need to include the correlation structure when choosing how wiggly or smooth the fitted trend should be. For full marks in regression I would expect them to give me a regression equation for either the GLS terms or the additive model in terms of smooth functions and bonus points for giving distributional assumptions including \( \sigma^2 \Lambda \) where \( \Lambda \) is a correlation matrix and residuals are zero mean normal or
Gaussian random variables.

A good answer would give clear reasoned justification for choice of model, appropriate consideration of the assumptions of the modelling approach they choose, plus appreciation of the need for model checking of residuals.

“c” could be investigated in a structural change context. CUSUM or F-stats/tests could be used to detect structural change or deviation from a model of constant linear trend. The student should appreciate that the data could be modelled as a linear decreasing trend in the first part of the record and no change/null model in the second half of the record.

For any of the time series the student shouldn’t advocate fitting every model or analysis we taught them. Targeted analysis using a key technique is best. Exception is testing for a non-linear trend and/or presence of structural change. Without more knowledge of the system in “d” either is a plausible model of the non-linear change over time in the annual level of the lake. Extra marks if the student appreciates this idea.